MTH 170: Foundations in Contemporary Mathematics

**Fair Division**

**Goal:** To divide an object or set of objects among a group of people in such a way that everyone gets at least their fair share.

**Rules:**
- The parties cannot communicate with each other.
- The parties are not allowed to know the other parties’ values on the object(s).
- The parties act in their best interest and avoid making emotional decisions.
- The fair division method should be able to be done between the parties without any outside intervention.

**Methods:**
- **Divider-Chooser Method**
  - “You cut, I choose” method.
  - Used for splitting evenly between only two parties.
  - The party designated as the divider splits the item(s) into two pieces that the party views as equal in value.
  - The other party designated as the chooser selects one of the two pieces.
  - The remaining piece goes to the divider.
  - **Example:**
    - Tom and Beth are splitting up a pizza between the two of them.
    - The pizza is half cheese and half pepperoni.
    - Tom likes cheese pizza twice as much as the pepperoni pizza. Beth likes both equally.
    - The two decide that Tom will be the divider, making Beth the chooser.
    - When Tom goes to divide the pizza, he must choose a division that will result in both parties getting a fair share in Tom’s eyes. Even though Tom values the cheese pizza more, Tom must divide the pizza in a way that he values as a fair share. If the two pieces are uneven, Tom might not get the piece he values as his fair share.
      - For example, Tom could split the pizza so that one piece was only cheese and the other piece was only pepperoni, but if Tom was left with the piece of just pepperoni pizza, he would get a piece that he does not want as much and does not value as a fair share in his eyes.
    - To guarantee that he gets a fair share, he can split the pizza so that both pieces have an equal amount of both cheese and pepperoni on each piece (both pieces are half cheese and half pepperoni).
    - After Tom splits the pizza into two pieces, Beth would choose one of the pieces and Tom would receive the remaining piece.

- **Lone Divider Method**
  - Used for splitting evenly among more than two parties.
  - The divider splits the item(s) into \( N \) equal pieces.
  - Each of the choosers places a declaration or bid on the pieces they want or view as at least a fair share.
  - If each chooser puts a declaration on different pieces, then those pieces can be distributed among the choosers and the divider will receive the remaining piece.
  - If two or more choosers place a declaration on the same piece, a non-contested piece will go to the divider, the remaining pieces are put together and the remaining players repeat the lone divider method with the remaining pieces (or divider-chooser method if only two remain).
Example:
- Marty, Sue, Frank, and Jimmy are trying to split a sandwich into four even pieces and decide to do so through the lone divider method. The four decide that Jimmy will be the divider.
- Jimmy will cut the sandwich in a way which he views to be a fair share for each of the four of them or 25% of the sandwich. The other two will put values on the three pieces as follows.

<table>
<thead>
<tr>
<th>Piece</th>
<th>Piece 1</th>
<th>Piece 2</th>
<th>Piece 3</th>
<th>Piece 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marty</td>
<td>40%</td>
<td>20%</td>
<td>30%</td>
<td>10%</td>
</tr>
<tr>
<td>Sue</td>
<td>35%</td>
<td>15%</td>
<td>35%</td>
<td>15%</td>
</tr>
<tr>
<td>Frank</td>
<td>20%</td>
<td>20%</td>
<td>20%</td>
<td>40%</td>
</tr>
<tr>
<td>Jimmy</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
</tr>
</tbody>
</table>

- Each person or party will declare any piece they believe is at least a fair share. Jimmy will receive any uncontested piece since he was the divider. Marty, Sue, and Frank will declare on pieces that they believe are at least a fair share.
  - Marty: Piece 1, Piece 3
  - Sue: Piece 1, Piece 3
  - Frank: Piece 4
- Frank is the only person to declare Piece 4. Therefore, he will get Piece 4.
- Pieces 1 and 3 are contested by Marty and Sue.
- Since Piece 2 is uncontested, Jimmy will take that piece.
- Marty and Sue then put piece 1 and 2 together and split them using the divider-chooser method since there are only two players remaining that haven’t received a piece.

Last Diminisher Method
- Used for splitting evenly among more than two parties.
- The parties are assigned an order randomly (pulling name out of a hat, drawing straws, etc.).
- The first player cuts a piece viewed as a fair share.
- Each party examines the piece in order, and cuts the piece (or diminishes the piece) if the piece is too big, or passes the piece to the next party in the order if the piece is too small.
- After the piece has been examined by each of the parties, the last person to cut or diminish the piece will take it and leave.
- This process continues until two parties remain and divider-chooser method can be used.

Example:
- Four people want to split a $20 cake evenly amongst themselves. They decide to split it up using the last diminisher method. The four decide to randomly choose an order in which each piece will be passed. The first player will cut a piece and pass it to the second. The table below shows the value of the piece according to the player when that player is handed the piece.

<table>
<thead>
<tr>
<th>Player</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>$6.00</td>
<td>$4.00</td>
<td>$5.50</td>
</tr>
</tbody>
</table>

- Player 2 receives the piece and values it as more than a fair share ($5.00). Player 2 will diminish that piece and pass it down to Player 3. Player 3 thinks the piece is too small now and will just pass the piece to Player 4. Player 4 thinks the piece is more than a fair share and will diminish the piece again.
• Player 4 will receive the piece because Player 4 was the last one to diminish the piece. Player 4 takes that piece then leaves. The remaining three players will use last diminisher again. Player 1 cuts a second piece and passes it to Player 2 like before.

<table>
<thead>
<tr>
<th>Player</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>$4.00</td>
<td>$3.00</td>
</tr>
</tbody>
</table>

• Player 2 is first to examine the piece again and will pass the piece because Player 2 values that piece as less than a fair share. Player 3 will also pass the piece for the same reason. Because none of the players diminished the piece, the player who cut the piece will take it. Therefore, Player 1 will take the piece and leave.

• The remaining two players can use divider-chooser to divide up what is left of the cake.

- **Moving Knife Method**
  - A referee slowly moves a “knife” across the object(s) until the first players calls STOP. The referee cuts off that piece and gives it to the player.
  - This repeats until there is one player left who receives whatever is remaining from the object(s).
  - **Example:**
    - Suppose that four salespeople decided to use this approach to divide Washington. Rather than move the “knife” from left to right, they decide to move it from top to bottom.
    - The referee starts moving a line down a map of the state. Henry is the first to call STOP when the knife is at the position shown, giving him the portion of the state above the line.
    - Marjo is the next to call STOP when the knife is at the position shown, giving her the second portion of the state.
    - Bob is the next to call STOP, leaving Beth with the southernmost portion of the state.
Sealed Bids Method

- This method is to help divide a set of items that cannot be divided into pieces (house, car, furniture, etc.).
- Each party secretly lists a value they believe each item is worth, also known as their sealed bid.
- Each party determines what a fair share is to them by dividing the total value they placed on all items by the number of parties.
- The sealed bids are collected and the highest bidder on each of the items will get that item.
- The value of all items received is totaled for each party.
- If a player received more than their fair share, they pay the difference to a holding pile.
- Any player who received less than their fair share is paid the difference from the holding pile.
- Any surplus or leftover money in the holding pile is paid evenly among the parties.

Example:

- Two siblings, John and Susan, need to split up several items amongst themselves and decide to do so using sealed bids.
- The two place bids on the three items and count them up. This will also determine what each of them values a fair share to be.

<table>
<thead>
<tr>
<th></th>
<th>John</th>
<th>Susan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item 1</td>
<td>$20</td>
<td>$30</td>
</tr>
<tr>
<td>Item 2</td>
<td>$50</td>
<td>$40</td>
</tr>
<tr>
<td>Item 3</td>
<td>$80</td>
<td>$90</td>
</tr>
<tr>
<td>Total</td>
<td>$20 + $50 + $80 = $150</td>
<td>$30 + $40 + $90 = $160</td>
</tr>
<tr>
<td>Fair Share</td>
<td>$150 / 2 = $75</td>
<td>$160 / 2 = $80</td>
</tr>
</tbody>
</table>

- The highest bidder for each item will receive that item. Therefore, John gets item 2, and Susan gets items 1 and 3.
- Susan received items 1 and 3, but the total value of the two items is $30 + $90 = $120. This is more than a fair share to Susan and she must pay the difference of $120 − $80 = $40 into a holding pile.
- John only received item 2 which is worth less than a fair share to him. John will receive money from the holding pile to make up for the difference. John will receive $75 − $50 = $25 from the holding pile.
- The $15 remaining in the holding pile is then distributed evenly between the two. Both siblings receive $7.50 from the holding pile.
- In the end, John received item 2 and received $32.50. Susan received items 1 and 3 and paid $32.50 to John.